VIEW FROM THE PENNINES: EULER GOES BALLISTIC

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It is easy to construct theories about why something happens, and even to start to believe them. My early visits to the Pennine reservoirs convinced me that an interesting nineteenth century phenomenon must have been the shift from the church to engineering as a destination of choice for impoverished yet well-born younger sons. This theory was based on two observations. The obsessive levels of detail in small bridges crossing isolated catchments (conduits) and linking one windswept piece of moorland to another equally windswept and equally bereft of human occupancy; and the obvious training in classical mythology which led water engineers to construct vertical cylindrical outlet towers which, when operating, were surely intended to mimic the whirlpools of Charybdis.

What middle-class-ocentric tosh! The younger sons probably went to the colonies where they could live appropriately on the backs of others without troubling their brains, and the water engineers were a peripatetic band of experts, who learned their trade through apprenticeship and may well have started as stone-masons or mill-wrights (see [5] but don't believe everything you read!).

So, don't accept anything just because it sounds nice – in theory, theory works in practice, but in practice, practice can be a lot more complicated (to paraphrase a quote I cannot place). Galileo was responsible for another nice theoretical idea which is still current: the parabolic projectile trajectory. This is so neat that it is taught at both school and undergraduate level as a triumph of the Newtonian revolution, giving practical advice on how to aim a cannon or catch a cricket ball through a magical mathematical formula.

What starry-eyed tosh! Galileo's intuition may have come from celestial observations where wind resistance is not an issue, but whether firing cannon or hitting cricket balls this air resistance and its effects cannot be ignored. However, the beauty and simplicity of the mathematics has meant that even though an experienced bombardier would have known how inaccurate they were, tables for cannon ranges based

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on the parabola were printed and presumably distributed for use in the seventeenth century. It is worth recalling a bit of this theory and some consequences. If x and z are horizontal and vertical coordinates respectively, then the only force assumed to act on a particle is a constant gravitational force giving acceleration $-g$. If (u, w) are the particle's horizontal and vertical components of velocity with initial values (U, V) then there is no acceleration in the x direction so $u = U$ and $\frac{df}{dt} = U\frac{df}{dx}$ for differentiable functions f. Hence the vertical equations of motion, $\frac{d^2z}{dt^2} = -g$, can be rewritten in terms of the dependence of the height on x as $U^2\frac{d^2z}{dx^2} = -g$ from which two integrations with initial position $(0, 0)$ give

$$z = -\frac{gx^2}{2U^2} + \frac{Vx}{U} \tag{1}$$

the celebrated parabolic arc. In the early days of ballistics, the immediate application on most people's minds was the trajectory of a cannonball or a musket ball. An English mathematician, Benjamin Robins (1707-1751), was one of the most influential early researchers in this area. He introduced new experimental methods to determine the speeds and trajectories of musket balls, and combined this with theoretical insights. He made it very clear that Galileo's model needed modification. As Steele puts it in [6] 'According to parabolic theory, a 24-pound solid shot fired from a canon could reach 16 miles, should its actual initial velocity be used. In practice, its maximum range was less than 3 miles because of air resistance.' Robins was only able to make these comparisons because he was able to measure initial muzzle velocities.

Jean Bernoulli had already considered the mathematics of this problem, using the assumption (due to Newton amongst others) that a force opposing the motion proportional to the square of the velocity needs to be added to the gravitational force. This leads to the vector equations

$$\ddot{\mathbf{r}} = -g\hat{\mathbf{z}} - k|\dot{\mathbf{r}}|^2\hat{\dot{\mathbf{r}}} \tag{2}$$

where hats denote unit vectors and dots differentiation with respect to time. In coordinate form this is

$$\begin{aligned} \ddot{x} &= -k(\dot{x}^2 + \dot{z}^2)^{\frac{1}{2}}\dot{x} \\ \ddot{z} &= -g - k(\dot{x}^2 + \dot{z}^2)^{\frac{1}{2}}\dot{z} \end{aligned} \tag{3}$$

This is certainly more realistic, but poses a different technical challenge as it is a set of coupled nonlinear differential equations. If the acceleration due to gravity is ignored the equations are relatively easy to solve, and solutions lie on a straight line, so this appears even less realistic than the parabolic arc. Clearly both terms of the second equation need to be kept (although we could try to glue together an initially approximate straight line solution – high velocity – to a parabolic turning point – low velocity). This is where Euler came in. He produced a translation of Robins' book, *New Principles of Gunnery* (1742), and in characteristic fashion added a commentary which extended the volume from 150 pages to over seven hundred! He considered the trajectory of a projectile explicitly in an article of 1753 [2]. Realising that he was not going to be able to solve (3) analytically, he developed approximation methods to determine approximate solutions, although he didn't actually tabulate many results. This practical work was left to Robins to complete in his turn [6].

Before leaving this model there is one further observation which is easy to make. Transforming to polar coordinates in the velocity (p, ϕ) with $(u, w) = (p \cos\phi, p \sin\phi)$, (3) is

$$\dot{p} = -g \sin\phi - kp^2, \quad \dot{\phi} = -gp^{-1} \cos\phi \tag{4}$$

and from the angular equation consideration of the sign of the right hand side shows that provided the particle is fired off a cliff and can stay in the air sufficiently long, then $\phi \to -\frac{\pi}{2}$, i.e. the particle approaches a vertical descent. Substituting this into the radial equation shows that in this descent $p \to \frac{g}{k}$, the terminal velocity for the problem. Overall, though, the message is that the effect of the wind resistance on a cannonball (or equivalent object) fired at high velocity is to make the initial part of the trajectory closer to a straight line, and the final part rather steeper than would be expected from the parabolic equation, and this explains why the parabolic arc overestimates the range of the trajectory.

Euler makes an interesting comment on the assumptions which build a wind resistance of this form. He says ([2], p. 326, my translation)

> This formula [for the constant of proportionality of the resistance force] will hold when the movement of the ball is not too fast so that the air can quite freely fill the space which the ball has left behind. But if the movement is so rapid that the air is unable to occupy instantaneously the space which the ball had occupied, so that this space remains empty, at least for an instant, then the front part of the ball is subject to the atmospheric pressure which, not being balanced by an equal pressure behind, it is clear that the resistance will be increased by the entire atmospheric pressure on the anterior of the ball.

This insightful comment leads to the next, and more modern, approach to ballistics: the study of fluid (air) flow over the surface of the bullet or cannonball.

It has long been understood that fluid dynamics would be needed to predict the motion of objects through the air, although I suspect it would surprise those early researchers to learn just how hard this can be. Robins had already observed the bending of a musket ball path due to its spin. This is the Magnus effect, which is sometimes, probably more properly, referred to as the Robins effect: spin about an axis perpendicular to the motion creates an effective force which is in the direction of the vector product of the spin axis and the direction of relative wind motion due to the change in pressure on the sides: at one side (the right say) the rotation is in the direction of the relative wind motion, i.e. in the opposite direction to the motion of the ball, and hence the air speed is greater, on the other side (the left) the effect is the opposite, so the pressure difference created makes the ball move towards the side moving with the wind, or against the direction of motion (the right here). Robins observed this by bending the barrel of a musket, thus imparting spin to the ball, and firing through an array of tissue paper, so that the path could be deduced from the sequence of holes [6].

Even in the absence of spin, the problem of a sphere moving through a fluid is horrendously complicated and depends on the behaviour of boundary layers around the ball. Drag is usually characterized through a *drag coefficient*, C_D, which depends on the Reynolds number Re of the flow, where for a sphere of radius a moving with speed U in a fluid (e.g. the air) with viscosity ν, $Re = 2aU/\nu$. The drag force of a projectile with projected area A in the direction of motion due to a fluid of density ρ is then

$$F = \tfrac{1}{2}C_d\rho U^2 A \tag{5}$$

so in the case of a sphere with $A = \pi a^2$ this becomes

$$F = \tfrac{1}{2}C_d\pi a^2\rho U^2 \tag{6}$$

Of course, defining the drag coefficient does not solve any problems, but drag force is generally expressed in terms of the drag coefficient rather than the force itself. The exception is probably the Stokes force, due to the idealized smooth flow of a fluid around a sphere, and which has $F = 6\pi\nu aU$. A comparison between this, (6) and the definition of the Reynolds number shows

$$C_D = 24/Re \tag{7}$$

in this case. As the Reynolds number increases, the flow separates from the sphere (one possible interpretation of the extended quote from Euler), and eddies are formed at the downstream end of the sphere. Separation initially occurs close to the stagnation point at the back of the sphere, but moves out as Re increases, with C_D reaching a value of about unity. Then, at about $Re \sim 5 \times 10^5$ or more, the flow behind the sphere becomes turbulent, and there is a dramatic decrease in C_D (see e.g. the experiments reported in [1]). This is one reason for the dimples in golf balls: the dimples help create a turbulent wake, thus reducing drag. Note that other reasons may also be relevant, such as maximizing the Magnus effect of spin imparted on the ball to give lift rather than sideways motion.

A more interesting interpretation of the Euler quote is the possibility that he was thinking about supersonic motion, where the speed of the projectile is faster than the speed of sound in the fluid, the speed at which pressure waves propagate. Here the drag coefficient almost triples with the development of shocks and uses arguments closer to special relativity and electrodynamics than standard fluid dynamics. The effect was (surprise, surprise) known to Robins; he even recognised a possible link to the speed of sound noting 'the velocity, at which the moving body shifts its resistance, is nearly the same, with which sound is propagated through the air.' (quoted in [6] from Robins' collected works), however, he was unable to establish a theoretical explanation for this link.

Part of the shock structure is easy to understand. Consider a particle moving with velocity v along the $x-$axis in the (x, y) plane through a fluid with sound speed c, $c < v$. Choose coordinates such that the particle is at $x = 0$ at time $t = 0$ then it was at $X = vt$ ($X < 0$) at time $t < 0$. A pressure signal can have radiated out a maximum distance $c|t|$ so as to lie on a circle centred at $(X, 0)$ with radius $c|t| = cX/v$, or in terms of $t < 0$.

$$(x - vt)^2 + y^2 = c^2t^2 \tag{8}$$

The set of points in the plane which can feel these pressure waves is thus the union of these circles for all $t < 0$, and those that cannot yet feel the influence of the supersonic particle's motion are separated from these by a curve which is the caustic created by the circles. In fact, there is a pair of lines $y = \pm qx$ ($x < 0$) which are tangential to every such circle and which thus forms this boundary. To determine $q > 0$ note that if the line intersects the circle centred at $Z = (vt, 0)$ tangentially at P, then $|ZP| = |ct|$ and OPZ is a right angled triangle with right angle at P. Hence the length of the hypoteneuse is $|vt|$ and if

θ is the angle at O of this triangle then $\sin\theta = c/v$. The ratio $M = v/c$ is called the Mach number, so $\sin\theta = M^{-1}$. Clearly

$$q = \tan\theta = 1/\sqrt{M^2 - 1}. \tag{9}$$

To a first approximation there is a shock across this line where the pressure and density of the fluid can jump. In fact things are inevitably a bit more complicated with, typically, at least two different shocks, one at the front and one at the back of the projectile, but a full analysis of even the approximate equations describing this would take too long to go into here. Miller and Bailey [4] give a fascinating account of what can be gleaned from nineteenth century experiments on the drag coefficient in both subsonic and supersonic cases.

There are two morals to this story. First, that the study of projectiles is much more difficult than the account generally given in elementary courses. Indeed, even the effects described here have subtleties I haven't attempted to include (for example at low Reynolds numbers there can be a reverse Magnus effect, and bullets do not tend to be oriented in the direction of their motion but have a yaw) or to look at the drag coefficient in the supersonic case. At subsonic speeds boundary layer separation can be further complicated by cross-winds which give other dynamical effects. Rotation, even at low Reynolds numbers requires sophisticated numerical techniques [3]. These effects are all understood through continuum models, but even the simplest non-continuum models of air resistance cannot be solved explicitly.

The second moral is that once again Euler's contributions foreshadow many of the problems which are still active issues in mathematics. In this case there is a second tercentenary to add to that of Euler, and which should also be celebrated this year. Like Euler, Benjamin Robins was born in 1707. He developed experimental apparatus which allowed him to measure important ballistic quantities and observe many of the effects which remain hard to explain today. It is true that he contributed to an imperial war effort, but then so did Euler. Today they might both have been interesting scientific guests at a game of cricket or baseball, though I suspect that both would also have remained aware of the broader implications of their work even so.

References

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